Kähler Normal Coordinate Expansion in Supersymmetric Theories

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Abstract

The Riemann normal coordinate expansion method is generalized to a Kähler manifold. The Kähler potential and holomorphic coordinate transformations are used to define normal coordinates preserving the complex structure. The existence of these Kähler normal coordinate is shown explicitly to all orders. The formalism is applied to background field methods in supersymmetric nonlinear sigma models.

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1 Introduction

In nonlinear sigma models, a field variable $\varphi(x)$, defined at a space-time point x, takes a value on a Riemannian manifold called the target manifold. Its S-matrices are invariant under an arbitrary field redefinition which corresponds to a general coordinate transformation in the target manifold. In the perturbation theory, we assume that configurations of field variables are very close to a background field φ_0 corresponding to a vacuum, and expand the Lagrangian as a power series in $\delta\varphi^i = \varphi^i - \varphi_0^i$ to define the interaction Lagrangian. In general, the expansion coefficients of this power series are non-covariant quantities, like the Christoffel symbols $\Gamma^i{}_{jk}$. It is very convenient to choose a special coordinate system called *the Riemann normal coordinates*, in which all the expansion coefficients are covariant tensors. In this coordinate system, results of the perturbation theory are expressed in terms of covariant quantities, and reparameterization invariance becomes manifest. This is one reason why Riemann normal coordinates are widely used in the renormalization of sigma models [1]. The Riemann normal coordinates around φ_0 are usually defined as a coordinate system in which all geodesics passing through φ_0 are straight lines, and neighboring points are identified with tangent vectors at φ_0.

Generally speaking, we have to solve the geodesic equation in order to find the coordinate transformation to the Riemann normal system. In this article, we propose a simple alternative algorithm to find the coordinate transformation to the normal coordinate system in the case that the nonlinear sigma models have $N = 2$ supersymmetry in two dimensions. The existence of $N = 2$ supersymmetry in two dimensions requires the target manifold to be a Kähler manifold [2]. Instead of using a metric, we rely heavily on the Kähler potential $K(\varphi, \varphi^*)$, which fixes the geometry of the Kähler manifold, to transform to Riemann normal coordinates. We call our normal coordinates "Kähler normal coordinates".[1] One of novel features of our method is that we do not need discussions of geodesics, in contrast to the real Riemann manifold [4].

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[1] Although Clark and Love [3] presented one such method, it requires an isometry of the target manifold. On the other hand, our method is valid for any Kähler manifold.

Nonlinear sigma models with $N = 2$ supersymmetry are very important tools to describe superstring theory in compactified space-time. Recently we found auxiliary field formulations for nonlinear sigma models on Hermitian symmetric spaces [5]. We hope our new method together with the auxiliary field formulation will play a significant role in the non-perturbative analyses of these models.

One can also apply the normal coordinate method to nonlinear sigma models in four dimensions. The $N = 1$ supersymmetry in four dimensions has the same structure as the $N = 2$ supersymmetry in two dimensions. (The latter is a direct dimensional reduction of the former.) $N = 1$ supersymmetric nonlinear sigma models in four dimensions appear as low-energy effective theories describing (quasi-)Nambu-Goldstone bosons when global symmetry is spontaneously broken with preserved supersymmetry [6]. The low-energy theorems of two-body scattering amplitudes of these bosons are discussed in Ref. [7], where a Kähler normal coordinate expansion to fourth order is used. An expansion to higher orders is necessary for calculations of many-body scattering amplitudes.

This paper is organized as follows. In section 2 we construct the Kähler normal coordinate expansion and present a theorem asserting that all the coefficients are covariant. This method is applied to supersymmetric nonlinear sigma models in section 3. We summarize the geometry of the Kähler manifold in Appendix A. A proof of the theorem is given in Appendix B.

2   Kähler normal coordinate

In the case of real Riemannian manifolds, we have to solve the geodesic equation to find the transformation to the Riemann normal coordinates. We propose to use the Kähler potential to find the holomorphic transformation to the Kähler normal coordinates. If we expand the Kähler potential in a Taylor series about the origin, the coefficients are in general non-covariant quantities. After an appropriate coordinate transformation, all coefficients are expressed in terms of covariant quantities. In Subsec. 2.1 we discuss the Kähler normal coordinates to fourth order. All non-covariant terms in the Taylor expansion of the Kähler potential are eliminated

explicitly to this order. In Subsec. 2.2, we generalize the coordinate transformation to all orders and give a theorem ensuring the covariance of all coefficients in the new coordinate system. A proof of the theorem is given in Appendix B, and we calculate several lower order coefficients to confirm their covariance in the remainder of Subsec. 2.2.

2.1 The Fourth order

In this subsection, we discuss the fourth order Kähler normal coordinates. Let (z^i, z^{*i}) be the general complex coordinate of a patch of a Kähler manifold. A Kähler manifold is characterized by a Kähler potential $K(z, z^*)$, which is defined in each coordinate patch of the manifold. Then the Kähler metric is given by

$$g_{ij^*}(z, z^*) = \partial_i \partial_{j^*} K(z, z^*), \tag{2.1}$$

where the differentiations are with respect to the coordinates z^i and z^{*j}. The metric is invariant under the Kähler transformation

$$K(z, z^*) \rightarrow K(z, z^*) + f(z) + f^*(z^*). \tag{2.2}$$

Geometric quantities, such as the connection and the curvature, can be calculated from the metric and hence from the Kähler potential, as summarized in Appendix A.

We frequently use convenient notation for partial derivatives. In this notation, indices preceded by a comma denote derivatives. For example, the definition of the Kähler metric (2.1) is written $g_{ij^*}(z, z^*) = K_{,ij^*}$. Then the Taylor expansion of the Kähler potential around the coordinate origin $z^i = 0$ is

$$
\begin{aligned}
K(z, z^*) &= \sum_{N,M=0}^{\infty} \frac{1}{N!M!} K_{,i_1 \cdots i_N j_1^* \cdots j_M^*}|_0 z^{i_1} \cdots z^{i_N} z^{*j_1} \cdots z^{*j_M} \\
&= K|_0 + F(z) + F^*(z^*) \\
&\quad + g_{ij^*}|_0 \, z^i z^{*j} + \frac{1}{2} \Gamma_{i^*jk}|_0 \, z^{*i} z^j z^k + \frac{1}{2} \Gamma_{ij^*k^*}|_0 \, z^i z^{*j} z^{*k} \\
&\quad + \frac{1}{4} (R_{ij^*kl^*} + g_{mn^*} \Gamma^m{}_{ik} \Gamma^{n^*}{}_{j^*l^*})|_0 \, z^i z^k z^{*j} z^{*l} \\
&\quad + \frac{1}{6} \partial_k \Gamma_{l^*ij}|_0 \, z^i z^j z^k z^{*l} + \frac{1}{6} \partial_{k^*} \Gamma_{li^*j^*}|_0 \, z^{*i} z^{*j} z^{*k} z^l + O(5), \tag{2.3}
\end{aligned}
$$

where

$$F(z) = \sum_{N=1}^{\infty} \frac{1}{N!} K_{,i_1\cdots i_N}|_0 z^{i_1} \cdots z^{i_N} = K_{,i}|_0\, z^i + \frac{1}{2} K_{,ij}|_0\, z^i z^j + \cdots \qquad (2.4)$$

is holomorphic and can be eliminated by the Kähler transformation (2.2). The expansion coefficients are identified with the connection $\Gamma^i{}_{jk}$ or the curvature tensor $R_{ij^*kl^*}$ by using the definition of these geometrical quantities summarized in Appendix A. The subscripts $|_0$ indicate that the values in question are evaluated at the origin $z^i = 0$. (We sometimes omit "0" when the expansion point is obvious.) With the exception of g_{ij^*} and $R_{i^*jk^*l}$, all coefficients are non-covariant to this order. A holomorphic coordinate transformation to eliminate these non-covariant quantities is found without difficulty. Note that Eq. (2.3) can be rewritten as

$$
\begin{aligned}
K(z, z^*) &= K| + F(z) + F^*(z^*) \\
&+ g_{mn^*}\left(z^m + \frac{1}{2}\Gamma^m{}_{jk}|z^j z^k + \frac{1}{6} g^{ml^*}\partial_k \Gamma_{l^*ij}|z^i z^j z^k\right) \\
&\quad \times \left(z^n + \frac{1}{2}\Gamma^n{}_{op}|z^o z^p + \frac{1}{6} g^{nr^*}\partial_q \Gamma_{r^*op}|z^o z^p z^q\right)^* \\
&+ \frac{1}{4} R_{i^*jk^*l}|z^{*i} z^{*k} z^j z^l + O(5).
\end{aligned}
\qquad (2.5)
$$

So by the *holomorphic* coordinate transformation

$$\omega^i = z^i + \frac{1}{2}\Gamma^i{}_{jk}|z^j z^k + \frac{1}{6} g^{im^*}\partial_l \Gamma_{m^*jk}|z^j z^k z^l, \qquad (2.6)$$

it can be written as

$$K(\omega, \omega^*) = K| + \tilde{F}(\omega) + \tilde{F}^*(\omega^*) + g_{ij^*}|\omega^i \omega^{*j} + \frac{1}{4} R_{i^*jk^*l}|\omega^{*i}\omega^{*k}\omega^j\omega^l + O(5), \quad (2.7)$$

where $\tilde{F}(\omega) \overset{\text{def}}{=} F(z(\omega))$. This coordinate transformation is invertible to give $z^i = z^i(\omega) = \omega^i - \frac{1}{2}\Gamma^i{}_{jk}|\omega^j\omega^k + \cdots$.

Non-covariant quantities remain in $\tilde{F}(\omega)$. Since the transformation (2.6) is *holomorphic*, $\tilde{F}(\omega)$ is still holomorphic and can be eliminated by a Kähler transformation (2.2). Therefore, all the expansion coefficients are expressed in terms of covariant quantities in the new coordinate system spanned by ω. This coordinate system is the desired Kähler normal coordinates. We can use normal coordinates defined about

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an arbitrary point $z^i{}_0$ by simply replacing z^i by $z^i - z^i{}_0$ in the above expression. All coefficients are evaluated at $z^i{}_0$ in this case.

It is useful to calculate some geometric quantities in these Kähler normal coordinates. The metric is calculated as

$$g_{ij^*}(\omega, \omega^*) = K_{,ij^*}(\omega, \omega^*) = g_{ij^*}| + R_{ij^*kl^*}|\omega^k \omega^{*l} + O(3), \tag{2.8}$$

and the inverse metric follows from $g^{ij^*} g_{kj^*} = \delta^i_k$ as

$$g^{ij^*}(\omega, \omega^*) = g^{ij^*}| + R^{ij^*}{}_{kl^*}|\omega^k \omega^{*l} + O(3). \tag{2.9}$$

From Eq. (2.7), we find that the curvature tensor *at the origin* is simply

$$R_{ij^*kl^*}| = K_{,ij^*kl^*}|. \tag{2.10}$$

Because of the commutativity of the differentiation, we obtain nontrivial relations among components of the curvature tensor from this equation (see Eq. (A.7)):

$$R_{ij^*kl^*} = R_{kj^*il^*} = R_{il^*kj^*}. \tag{2.11}$$

Since these equations are covariant, they hold in any coordinate system.

2.2   Kähler normal coordinate to all orders

In this subsection we generalize the coordinate transformation (2.6) to all orders. We then give a theorem which states that coefficients in the new coordinates are covariant. (A proof is given in Appendix B.) We also explicitly express the sixth order coefficients in terms of the curvature and its covariant derivatives for definiteness. The simple Taylor expansion is again

$$
\begin{aligned}
K(z, z^*) &= K| + F(z) + F^*(z^*) + g_{ij^*}|z^i z^{*j} \\
&\quad + \sum_{N=3}^{\infty} \sum_{M=1}^{N-1} \frac{1}{M!(N-M)!} K_{,i_1 \cdots i_M j_1^* \cdots j_{N-M}^*} |z^{i_1} \cdots z^{i_M} z^{*j_1} \cdots z^{*j_{N-M}} \\
&= K| + F(z) + F^*(z^*) + g_{ij^*}|z^i z^{*j} + \frac{1}{2}\Gamma_{i^*jk}|z^{*i} z^j z^k + \frac{1}{2}\Gamma_{ij^*k^*}|z^i z^{*j} z^{*k} \\
&\quad + \frac{1}{4}K_{,i^*jk^*l}|z^{*i} z^{*k} z^j z^l + \frac{1}{6}\partial_k \Gamma_{l^*ij}|z^i z^j z^k z^{*l} + \frac{1}{6}\partial_{k^*}\Gamma_{li^*j^*}|z^{*i} z^{*j} z^{*k} z^l \\
&\quad + \frac{1}{12}K_{,mij^*kl^*}|z^m z^i z^k z^{*j} z^{*l} + \frac{1}{12}K_{,m^*ij^*kl^*}|z^i z^k z^{*m} z^{*j} z^{*l} \\
&\quad + \cdots. \tag{2.12}
\end{aligned}
$$

The expansion coefficients are expressed in terms of geometric quantities by using the formulas given in Appendix A. As a generalization of Eq. (2.6), we perform a coordinate transformation given by

$$
\begin{aligned}
\omega^i &= z^i + \sum_{N=2}^{\infty} \frac{1}{N!} g^{ij^*} K_{,i_1 \cdots i_N j^*} | z^{i_1} \cdots z^{i_N} \\
&= z^i + \sum_{N=2}^{\infty} \frac{1}{N!} g^{ij^*} \partial_{i_3 \cdots i_N} \Gamma_{j^* i_1 i_2} | z^{i_1} \cdots z^{i_N} \\
&= z^i + \frac{1}{2} \Gamma^i{}_{jk} | z^j z^k + \frac{1}{6} (\partial_l \Gamma^i{}_{jk} + \Gamma^i{}_{ml} \Gamma^m{}_{jk}) | z^j z^k z^l + \cdots
\end{aligned}
\tag{2.13}
$$

in order to eliminate terms of the form $z^{i_1} z^{*j_1} \cdots z^{*j_N}$ or $z^{i_1} \cdots z^{i_N} z^{*j_1}$ $(N \geq 2)$. We thus obtain the expansion

$$
\begin{aligned}
K(\omega, \omega^*) &= K| + \tilde{F}(\omega) + \tilde{F}^*(\omega^*) + g_{ij^*} | \omega^i \omega^{*j} \\
&\quad + \sum_{N=4}^{\infty} \sum_{M=2}^{N-2} \frac{1}{M!(N-M)!} K_{,i_1 \cdots i_M j_1^* \cdots j_{N-M}^*} | \omega^{i_1} \cdots \omega^{i_M} \omega^{*j_1} \cdots \omega^{*j_{N-M}} \\
&= K| + \tilde{F}(\omega) + \tilde{F}^*(\omega^*) + g_{ij^*} | \omega^i \omega^{*j} + \frac{1}{4} K_{,i^* j k^* l} | \omega^{*i} \omega^{*k} \omega^j \omega^l \\
&\quad + \frac{1}{12} K_{,mij^* kl^*} | \omega^m \omega^i \omega^k \omega^{*j} \omega^{*l} + \frac{1}{12} K_{,m^* ij^* kl^*} | \omega^i \omega^k \omega^{*j} \omega^{*l} \omega^{*m} \\
&\quad + \cdots,
\end{aligned}
\tag{2.14}
$$

where all differentiations are with respect to the new coordinates, ω. In the previous subsection, we found that the coefficients in the expansion (2.7) are covariant quantities. The following theorem is a generalization of this observation.

Theorem. All coefficients in the expansion (2.14) are covariant.

We call such a coordinate system ω the "Kähler normal coordinates to all orders". We prove this theorem in Appendix B. In this subsection, as an illustration, we explicitly express the first several coefficients in terms of the curvature and the covariant derivatives. We refer to a tensor with N holomorphic lower indices and M anti-holomorphic lower indices as an (N, M) tensor. Since we have eliminated terms of the form $\omega^{i_1} \cdots \omega^{i_N} \omega^{*j_1}$ by the holomorphic coordinate transformation (2.13), the connection $\Gamma_{j_1^* i_1 i_2}$ differentiated any number of times with respect to the holomorphic coordinates ω,

$$
K_{,i_1 \cdots i_N j_1^*} = \partial_{i_3} \cdots \partial_{i_N} \Gamma_{j_1^* i_1 i_2} = \partial_{i_3} \cdots \partial_{i_N} (g_{kj_1^*} \Gamma^k{}_{i_1 i_2}),
\tag{2.15}
$$

vanishes at the origin. This implies that $g_{ij^*,i_1\cdots i_N}| = 0$, and thus we find

$$\partial_{i_1}\cdots\partial_{i_N}\Gamma^i{}_{jk}| = 0. \tag{2.16}$$

Hence if all of the covariant derivatives, acting on any tensor T, are holomorphic or anti-holomorphic, they become ordinary derivatives with respect to the coordinates at the origin:

$$D_{i_1}\cdots D_{i_N}T| = \partial_{i_1}\cdots\partial_{i_N}T|. \tag{2.17}$$

In particular, we have very simple formulas for the curvature tensor:

$$D_{i_1}\cdots D_{i_N}R_{ij^*kl^*}| = K,_{i_1\cdots i_N ij^*kl^*}|, \qquad D_{j_1^*}\cdots D_{j_N^*}R_{ij^*kl^*}| = K,_{j_1^*\cdots j_N^* ij^*kl^*}|. \tag{2.18}$$

For example, the $(3,2)$ tensor

$$\begin{aligned}
&D_m R_{ij^*kl^*}\\
&= \partial_m R_{ij^*kl^*} - \Gamma^n{}_{mi}R_{nj^*kl^*} - \Gamma^n{}_{mk}R_{ij^*nl^*}\\
&= K,_{mij^*kl^*} + g^{op^*}g^{qn^*}K,_{p^*qm}\,K,_{oj^*l}\,K,_{n^*ik}\\
&\quad - g^{on^*}\left(K,_{omj^*l^*}\,K,_{n^*ik} + K,_{oj^*l^*}\,K,_{mn^*ik} + K,_{n^*mi}\,R_{oj^*kl^*} + K,_{n^*mk}\,R_{ij^*ol^*}\right) \tag{2.19}
\end{aligned}$$

satisfies

$$D_m R_{ij^*kl^*}| = K,_{mij^*kl^*}|. \tag{2.20}$$

In the calculation of Eq. (2.19), we have used the formula

$$\partial_k g^{ij^*} = -g^{im^*}g^{lj^*}K,_{m^*lk}. \tag{2.21}$$

The symmetry property of the curvature, (2.11), derived in the previous subsection, can be generalized. When all of the covariant derivatives acting on the curvature are (anti-)holomorphic, all (anti-)holomorphic indices of the tensor are symmetric as a result of Eq. (2.18):

$$D_{i_1}\cdots D_{i_a}\cdots D_{i_N}R_{ij^*kl^*} = D_{i_1}\cdots D_i\cdots D_{i_N}R_{i_aj^*kl^*}, \qquad \text{etc.} \tag{2.22}$$

Again these equations are covariant, and hence they hold in any coordinate system. This relation can also be shown by the relation $[D_i, D_j] = 0$ and the Bianchi identity.[2]

We now examine whether other terms in the expansion (2.14) are also covariant. For example, the lowest non-trivial coefficient is, $K,_{mn^*ij^*kl^*}$, of the $(3,3)$-type. To evaluate this term we calculate one of the $(3,3)$-tensors from Eq. (2.19):

$$
\begin{aligned}
&D_{n^*}D_m R_{ij^*kl^*} \\
&= \underline{K,_{mn^*ij^*kl^*}} - (g^{ot^*}g^{sp^*}g^{qr^*} + g^{op^*}g^{qt^*}g^{sr^*})K,_{st^*n^*}\,K,_{p^*qm}\,K,_{oj^*l}\,K,_{r^*ik} \\
&\quad + g^{ot^*}g^{sr^*}[K,_{n^*t^*sm}\,K,_{oj^*l}\,K,_{r^*ik} + K,_{t^*sm}\,K,_{n^*oj^*l}\,K,_{r^*ik} + K,_{t^*sm}\,K,_{oj^*l}\,K,_{n^*r^*ik}) \\
&\qquad + K,_{st^*n^*}\,(K,_{omj^*l^*}\,K,_{r^*ik} + K,_{oj^*l^*}\,K,_{mr^*ik} \\
&\qquad\qquad + K,_{r^*mi}\,R_{oj^*kl^*} + K,_{r^*mk}\,R_{ij^*ol^*})\,] \\
&\quad - g^{or^*}(K,_{n^*omj^*l^*}\,K,_{r^*ik} + K,_{n^*oj^*l^*}\,K,_{mr^*ik} + \underline{K,_{omj^*l^*}\,K,_{n^*r^*ik}} + K,_{oj^*l^*}\,K,_{n^*mr^*ik} \\
&\qquad + \underline{K,_{n^*r^*mi}\,R_{oj^*kl^*}} + \underline{K,_{n^*r^*mk}\,R_{ij^*ol^*}} + K,_{r^*mi}\,\partial_{n^*}R_{oj^*kl^*} + K,_{r^*mk}\,\partial_{n^*}R_{ij^*ol^*} \\
&\qquad + K,_{rn^*j^*}\,D_m R_{io^*kl^*} + K,_{rn^*l^*}\,D_m R_{ij^*ko^*}). \tag{2.23}
\end{aligned}
$$

Here, only the underlined terms survive at the origin. We thus obtain a covariant expression of the coefficient $(3,3)$,

$$
K,_{mn^*ij^*kl^*}\,| = D_{n^*}D_m R_{ij^*kl^*}| + g^{or^*} R_{o(j^*ml^*}R_{in^*k)_{\mathrm{h}}r^*}|, \tag{2.24}
$$

where $(\cdots)_{\mathrm{h}}$ denotes cyclic permutation with respect to the holomorphic indices. (For example, $A_{(ij^*kl)_{\mathrm{h}}} = A_{ij^*kl} + A_{lj^*ik} + A_{kj^*li}$.) Note that this expression is not unique. For example, it can also be expressed as

$$
K,_{mn^*ij^*kl^*}\,| = D_m D_{n^*} R_{ij^*kl^*}| + g^{or^*} R_{o(j^*ml^*}R_{in^*k)_{\mathrm{ah}}r^*}|, \tag{2.25}
$$

where $(\cdots)_{\mathrm{ah}}$ denotes cyclic permutation with respect to the anti-holomorphic indices. [3] The right-hand sides of Eqs. (2.24) and (2.25) are manifestly symmetric on

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[2] The Bianchi identity in a Riemann manifold is $D_{(m}R_{ij)kl} = D_m R_{ijkl} + D_j R_{mikl} + D_i R_{jmkl} = 0$. In a Kähler manifold this becomes $D_m R_{ij^*kl^*} + D_{j^*}R_{mikl^*} + D_i R_{j^*mkl^*} = D_m R_{ij^*kl^*} - D_i R_{mj^*kl^*} = 0$. Hence we obtain $D_m R_{ij^*kl^*} = D_i R_{mj^*kl^*}$. The commutativity, $[D_i, D_j] = 0$, follows from Footnote 3, below.

[3] These two expressions, Eqs. (2.24) and (2.25), are related by a formula valid for any tensor

either the anti-holomorphic or holomorphic indices, but not both. The expressions symmetric with respect to both the holomorphic and anti-holomorphic indices are

$$
\begin{aligned}
K,_{mn^*ij^*kl^*}| &= \frac{1}{3}[D_{(n^*}D_m R_{ij^*kl^*)_{\text{ah}}} + g^{or^*}R_{o(j^*ml^*}R_{in^*k)r^*}]| \\
&= \frac{1}{3}[D_{(m}D_{n^*}R_{ij^*kl^*)_{\text{h}}} + g^{or^*}R_{o(j^*ml^*}R_{in^*k)r^*}]|,
\end{aligned}
\tag{2.27}
$$

where $(\cdots)$ denotes cyclic permutation with respect to both the holomorphic and anti-holomorphic indices, applied independently.[4]

In summary, from Eqs. (2.18) and (2.27), the manifestly covariant expression of the Kähler normal coordinate expansion to sixth order can be written as

$$
\begin{aligned}
&K(\omega, \omega^*) \\
&= K| + \tilde{F}(\omega) + \tilde{F}^*(\omega^*) + g_{ij^*}|\omega^i\omega^{*j} + \frac{1}{4}R_{ij^*kl^*}|\omega^i\omega^k\omega^{*j}\omega^{*l} \\
&+ \frac{1}{12}D_m R_{ij^*kl^*}|\omega^m\omega^i\omega^k\omega^{*j}\omega^{*l} + \frac{1}{12}D_{m^*}R_{ij^*kl^*}|\omega^i\omega^k\omega^{*j}\omega^{*l}\omega^{*m} \\
&+ \frac{1}{24}D_n D_m R_{ij^*kl^*}|\omega^n\omega^m\omega^i\omega^k\omega^{*j}\omega^{*l} + \frac{1}{24}D_{n^*}D_{m^*}R_{ij^*kl^*}|\omega^i\omega^k\omega^{*j}\omega^{*l}\omega^{*m}\omega^{*n} \\
&+ \frac{1}{108}(D_{(n^*}D_m R_{ij^*kl^*)_{\text{ah}}} + g^{or^*}R_{o(j^*ml^*}R_{in^*k)r^*})|\omega^m\omega^i\omega^k\omega^{*j}\omega^{*l}\omega^{*n} + O(7).
\end{aligned}
\tag{2.28}
$$

By the same procedure, in principle, one can obtain covariant expressions of the expansion to any desired order. All the coefficients are guaranteed to be covariant by the theorem.

In the rest of this section, we give Kähler normal coordinate expansions of some geometric quantities. The general expression of the Kähler metric in the Kähler T,

$$
[D_A, D_B]T_{C_1\cdots C_n} = \sum_{a=1}^{n} R_{ABC_a}{}^D T_{C_1\cdots C_{a-1}DC_{a+1}\cdots C_n},
\tag{2.26}
$$

where Roman uppercase letters are used for both the holomorphic and anti-holomorphic indices. Note that $[D_i, D_j] = [D_{i^*}, D_{j^*}] = 0$ as a result of the Kähler property. Hence we can define a "normal ordering" by putting D to the right of D_* to obtain the unique expressions. We use this expression in our proof of the theorem.

[4]From this equation, we obtain the nontrivial identity $D_{(n^*}D_m R_{ij^*kl^*)_{\text{ah}}} = D_{(m}D_{n^*}R_{ij^*kl^*)_{\text{h}}}$. This can be also proved by the formula (2.26).

normal coordinates to all orders is

$$g_{ij^*}(\omega, \omega^*) = g_{ij^*}| + \sum_{N=1}^{\infty} \sum_{M=1}^{\infty} \frac{1}{N!M!} K_{,ij^*i_1\cdots i_N j_1^*\cdots j_M^*} |\omega^{i_1}\cdots\omega^{i_N}\omega^{*j_1}\cdots\omega^{*j_M}. \quad (2.29)$$

Note that $g_{ij^*,i_1\cdots i_n}| = g_{ij^*,j_1^*\cdots j_n^*}| = 0$. The manifestly covariant expression of the expansion of the metric to fourth order is

$$g_{ij^*}(\omega, \omega^*)$$
$$= g_{ij^*}| + R_{ij^*kl^*}|\omega^k\omega^{*l} + \frac{1}{2}D_m R_{ij^*kl^*}|\omega^m\omega^k\omega^{*l} + \frac{1}{2}D_{m^*}R_{ij^*kl^*}|\omega^k\omega^{*l}\omega^{*m}$$
$$+\frac{1}{6}D_n D_m R_{ij^*kl^*}|\omega^n\omega^m\omega^k\omega^{*l} + \frac{1}{6}D_{n^*}D_{m^*}R_{ij^*kl^*}|\omega^k\omega^{*l}\omega^{*m}\omega^{*n}$$
$$+\frac{1}{12}(D_{(n^*}D_m R_{ij^*kl^*)_{\mathrm{ah}}} + g^{or^*}R_{o(j^*ml^*}R_{in^*k)r^*})|\omega^m\omega^k\omega^{*l}\omega^{*n} + O(5). \quad (2.30)$$

The inverse metric in the normal coordinate expansion can be calculated order by order from the definition $g^{ij^*}g_{j^*k} = \delta^i_k$. The expansion to fourth order is

$$g^{ij^*}(\omega, \omega^*)$$
$$= g^{ij^*}| + R^{ij^*}{}_{kl^*}|\omega^k\omega^{*l} + \frac{1}{2}D_m R^{ij^*}{}_{kl^*}|\omega^m\omega^k\omega^{*l} + \frac{1}{2}D_{m^*}R^{ij^*}{}_{kl^*}|\omega^k\omega^{*l}\omega^{*m}$$
$$+\frac{1}{12}D_n D_m R^{ij^*}{}_{kl^*}|\omega^n\omega^m\omega^k\omega^{*l} + \frac{1}{6}D_{n^*}D_{m^*}R^{ij^*}{}_{kl^*}|\omega^k\omega^{*l}\omega^{*m}\omega^{*n}$$
$$-\frac{1}{4}g^{iq^*}g^{j^*p}(D_{(n^*}D_m R_{pq^*kl^*)_{\mathrm{ah}}} + g^{or^*}R_{o(q^*ml^*}R_{pn^*k)r^*} - g^{or^*}R_{oq^*(mn^*}R_{kl^*)pr^*})|$$
$$\times\omega^m\omega^k\omega^{*l}\omega^{*n} + O(5). \quad (2.31)$$

The expansion of the connection can be calculated as

$$\Gamma_{j^*ik}(\omega,\omega^*) = K_{,ikj^*}(\omega,\omega^*) = g_{ij^*,k}(\omega,\omega^*)$$
$$= R_{ij^*kl^*}|\omega^{*l} + D_m R_{ij^*kl^*}|\omega^m\omega^{*l} + \frac{1}{2}D_{m^*}R_{ij^*kl^*}|\omega^{*l}\omega^{*m}$$
$$+\frac{1}{2}D_n D_m R_{ij^*kl^*}|\omega^n\omega^m\omega^{*l} + \frac{1}{6}D_{n^*}D_{m^*}R_{ij^*kl^*}|\omega^{*l}\omega^{*m}\omega^{*n}$$
$$+\frac{1}{6}(D_{(n^*}D_m R_{ij^*kl^*)_{\mathrm{ah}}} + g^{or^*}R_{o(j^*ml^*}R_{in^*k)r^*})|\omega^m\omega^{*l}\omega^{*n} + O(4). \quad (2.32)$$

Note that each term has at least one anti-holomorphic factor, ω^*, and hence the holomorphic derivatives of the connection are zero at the origin. The curvature tensor can be calculated to second order from Eqs. (2.31) and (2.32):

$$R_{ij^*kl^*}(\omega, \omega^*)$$

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$$= R_{ij^*kl^*}| + D_m R_{ij^*kl^*}|\omega^m + D_{m^*} R_{ij^*kl^*}|\omega^{*m}$$

$$+\frac{1}{2}D_n D_m R_{ij^*kl^*}|\omega^n \omega^m + \frac{1}{2}D_{n^*} D_{m^*} R_{ij^*kl^*}|\omega^{*m}\omega^{*n}$$

$$+\frac{1}{3}(D_{(n^*}D_m R_{ij^*kl^*)_{\rm ah}} + g^{or^*}R_{o(j^*ml^*}R_{in^*k)r^*} - g^{or^*}R_{oj^*ml^*}R_{in^*kr^*})|\omega^m \omega^{*n}$$

$$+O(3). \tag{2.33}$$

3 Applications to supersymmetric nonlinear sigma models

$N = 1$ ($N = 2$) supersymmetry in four (two) dimensions requires the target manifold of nonlinear sigma models to be a Kähler manifold [2]. We first present the derivation appearing in Ref. [8] of the Lagrangian of supersymmetric nonlinear sigma models in the general coordinates. After a brief remark on a field redefinition, we apply Kähler normal coordinates to background field methods.

3.1 Review of the chiral model

Chiral superfields satisfying the constraint $\bar{D}_{\dot\alpha}\Phi = 0$ are given by

$$\Phi^i(x, \theta, \bar\theta) = \Phi^i(y, \theta) = \varphi^i(y) + \sqrt{2}\theta\psi^i(y) + \theta\theta F^i(y), \tag{3.1}$$

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta, \qquad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}}. \tag{3.2}$$

The general D-term Lagrangian of the chiral superfields can be written as

$$\mathcal{L} = \int d^4\theta K(\Phi, \Phi^\dagger), \tag{3.3}$$

where the Kähler potential K is a real function. To calculate the Lagrangian written in terms of component fields, we expand the Kähler potential as in Eq. (2.12):

$$K = \sum_{N,M=0}^{\infty} \frac{1}{N!M!}K_{,i_1\cdots i_N j_1^*\cdots j_M^*}|_0 \Phi^{i_1}\Phi^{i_2}\cdots\Phi^{i_N}\Phi^{\dagger j_1}\Phi^{\dagger j_2}\cdots\Phi^{\dagger j_M}. \tag{3.4}$$

We define

$$K_{NM} = \Phi^{i_1}\Phi^{i_2}\cdots\Phi^{i_N}\Phi^{\dagger j_1}\Phi^{\dagger j_2}\cdots\Phi^{\dagger j_M}. \tag{3.5}$$

Its D-term can be calculated as

$$
\begin{aligned}
{[K_{NM}]_{\mathrm{D}}} &= \frac{\partial^2 K_{NM}(\varphi, \varphi^*)}{\partial\varphi^i \partial\varphi^{*j}} F^i F^{*j} - \frac{1}{2}\frac{\partial^3 K_{NM}(\varphi, \varphi^*)}{\partial\varphi^i \partial\varphi^{*j}\partial\varphi^{*k}} F^i \bar\psi^j \bar\psi^k \\
&\quad -\frac{1}{2}\frac{\partial^3 K_{NM}(\varphi, \varphi^*)}{\partial\varphi^{*i}\partial\varphi^j\partial\varphi^k} F^{*i}\psi^j\psi^k + \frac{1}{4}\frac{\partial^4 K_{NM}(\varphi, \varphi^*)}{\partial\varphi^i\partial\varphi^j\partial\varphi^{*k}\partial\varphi^{*l}}\psi^i\psi^j\bar\psi^k\bar\psi^l \\
&\quad +\frac{\partial^2 K_{NM}(\varphi, \varphi^*)}{\partial\varphi^i\partial\varphi^{*j}}\partial_\mu\varphi^i\partial^\mu\varphi^{*j} + i\frac{\partial^2 K_{NM}(\varphi, \varphi^*)}{\partial\varphi^i\partial\varphi^{*j}}\bar\psi^j\bar\sigma^\mu\partial_\mu\psi^i \\
&\quad +i\frac{\partial^3 K_{NM}(\varphi, \varphi^*)}{\partial\varphi^i\partial\varphi^j\partial\varphi^{*k}}(\bar\psi^k\bar\sigma^\mu\psi^i)\partial_\mu\varphi^j. \tag{3.6}
\end{aligned}
$$

Here we have used the equations

$$
\begin{aligned}
K_{N0}(\Phi) &= K_{N0}(\varphi) + \sqrt{2}\theta\psi^i\frac{\partial K_{N0}(\varphi)}{\partial\varphi^i} \\
&\quad +\theta\theta\left(F^i\frac{\partial K_{N0}(\varphi)}{\partial\varphi^i} - \frac{1}{2}\psi^i\psi^j\frac{\partial^2 K_{N0}(\varphi)}{\partial\varphi^i\partial\varphi^j}\right), \tag{3.7}
\end{aligned}
$$

$$
\begin{aligned}
{[\Phi^i\Phi^{j\dagger}]_{\mathrm{D}}} &= F^i F^{*j} - \frac{1}{4}\varphi^i\Box\varphi^{*j} - \frac{1}{4}\Box\varphi^i\varphi^{*j} + \frac{1}{2}\partial_\mu\varphi^i\partial^\mu\varphi^{*j} \\
&\quad -\frac{i}{2}\partial_\mu\bar\psi^j\bar\sigma^\mu\psi^i + \frac{i}{2}\bar\psi^j\bar\sigma^\mu\partial_\mu\psi^i, \tag{3.8}
\end{aligned}
$$

and partial integration.[5] From Eqs. (3.4) and (3.6), the general Lagrangian of chiral superfields can be written as

$$
\begin{aligned}
\mathcal{L} &= g_{ij^*}F^i F^{*j} - \frac{1}{2}g_{im^*}\Gamma^{m^*}_{j^*k^*}F^i\bar\psi^j\bar\psi^k - \frac{1}{2}g_{mi^*}\Gamma^m_{jk}F^{*i}\psi^j\psi^k \\
&\quad +g_{ij^*}\partial_\mu\varphi^i\partial^\mu\varphi^{*j} + ig_{ij^*}\bar\psi^j\bar\sigma^\mu\partial_\mu\psi^i + ig_{lk^*}\Gamma^l_{ij}\bar\psi^k\bar\sigma^\mu\psi^i\partial_\mu\varphi^j \\
&\quad +\frac{1}{4}g_{ij^*,kl^*}\psi^i\psi^k\bar\psi^j\bar\psi^l. \tag{3.9}
\end{aligned}
$$

The equation of motion of F^i reads

$$
F^i = \frac{1}{2}\Gamma^i{}_{jk}(\varphi, \varphi^*)\psi^j\psi^k. \tag{3.10}
$$

By substituting this back into Eq. (3.9), we obtain the Lagrangian of the supersymmetric nonlinear sigma model in the component fields,

$$
\mathcal{L} = g_{ij^*}(\varphi, \varphi^*)\partial_\mu\varphi^i\partial^\mu\varphi^{*j} + ig_{ij^*}(\varphi, \varphi^*)\bar\psi^j\bar\sigma^\mu(D_\mu\psi)^i + \frac{1}{4}R_{ij^*kl^*}(\varphi, \varphi^*)\psi^i\psi^k\bar\psi^j\bar\psi^l,
\tag{3.11}
$$

---

[5] This partial integration can be carried out, since the coefficients of Eq. (3.4) are constant.

where D_μ on the fermion is a pull-back of the covariant derivative on target manifolds, where the fermion behaves like a tangent vector (see Eq. (3.15), below):

$$(D_\mu \psi)^i = \partial_\mu \psi^i + \partial_\mu \varphi^j \Gamma^i{}_{jk}(\varphi, \varphi^*)\psi^k. \tag{3.12}$$

3.2 Field redefinition of chiral superfields

Before proceeding to discussion of the Kähler normal coordinates of nonlinear sigma models, we discuss a field redefinition of chiral superfields as a general coordinate transformation on target manifolds. Since a holomorphic function of chiral superfields $\Phi^i(x, \theta, \bar{\theta})$ $(i = 1, \cdots, n)$ is a chiral superfield, new fields $\Phi'^i(x, \theta, \bar{\theta})$ defined by

$$\Phi'^i(x, \theta, \bar{\theta}) = f^i(\Phi^j(x, \theta, \bar{\theta})) \tag{3.13}$$

are chiral superfields and can be used as coordinates of the Kähler manifold. The right-hand side can be written in component fields from Eq. (3.7) as

$$f^i(\Phi(y, \theta)) = f^i(\varphi(y)) + \sqrt{2}\theta\psi^j \frac{\partial f^i(\varphi)}{\partial \varphi^j}(y)$$
$$+ \theta\theta \left(F^j \frac{\partial f^i(\varphi)}{\partial \varphi^j}(y) - \frac{1}{2}\psi^j \psi^k \frac{\partial^2 f^i(\varphi)}{\partial \varphi^j \partial \varphi^k}(y) \right). \tag{3.14}$$

The field redefinitions of the component fields are

$$\varphi'^i(x) = f^i(\varphi(x)),$$
$$\psi'^i(x) = \frac{\partial f^i(\varphi(x))}{\partial \varphi^k}\psi^j(x),$$
$$F'^i(x) = \frac{\partial f^i(\varphi(x))}{\partial \varphi^j}F^j(x) - \frac{1}{2}\frac{\partial^2 f^i(\varphi(x))}{\partial \varphi^j \partial \varphi^k}\psi^j \psi^k(x). \tag{3.15}$$

Note that the field dependences on x are the same as those on y, since the relation $y = x + i\theta\sigma\bar{\theta}$ includes θ and $\bar{\theta}$. The first equation represents a general coordinate transformation, whereas the second equation implies that the fermions transform as a tangent vector on the target manifold, as expressed by Eq. (3.12).

For later use, we point out the field definition (3.13) can be generalized to

$$\Phi'^i(y, \theta) = f^i(\Phi^j(y, \theta), \varphi_0(y)), \tag{3.16}$$

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where $\varphi_0(y)$ is an additional bosonic field. We consider φ_0 as a background field in the next subsection. Note that the bosonic field φ_0 can depend on y but not on x. This is because we can preserve chirality: $\bar{D}_{\dot{\alpha}}\Phi^i = 0$ implies $\bar{D}_{\dot{\alpha}}\Phi'^j = 0$, since the spinor derivative $\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}$ does not include y in the y-representation. Transformations of the component fields are given simply by

$$
\begin{aligned}
\varphi'^i(x) &= f^i(\varphi(x), \varphi_0(x)), \\
\psi'^i(x) &= \frac{\partial f^i(\varphi(x), \varphi_0(x))}{\partial\varphi^k}\psi^j(x), \\
F'^i(x) &= \frac{\partial f^i(\varphi(x), \varphi_0(x))}{\partial\varphi^j}F^j(x) - \frac{1}{2}\frac{\partial^2 f^i(\varphi(x), \varphi_0(x))}{\partial\varphi^j\partial\varphi^k}\psi^j\psi^k(x). \quad (3.17)
\end{aligned}
$$

The bosonic fields depending on y and x are related as

$$
\varphi_0(y) = \varphi_0(x) + \partial_\mu\varphi_0(i\theta\sigma^\mu\bar{\theta}) + \frac{1}{4}\Box\varphi_0(x)\theta\theta\bar{\theta}\bar{\theta}, \quad (3.18)
$$

and the difference between $\varphi_0(y)$ and $\varphi_0(x)$ contains at least a term proportional to θ and $\bar{\theta}$. The transformation (3.16) may depend on a bosonic field through an arbitrary tensor (or non-tensor) $T_{i_1\cdots j_1^*\cdots}(\varphi_0(y), \varphi_0^*(y))$ on the target manifold. It can be expanded around $\varphi_0(x)$ as

$$
\begin{aligned}
&T(\varphi_0(y), \varphi_0^*(y)) \\
=\ &T\left(\varphi_0(x) + \partial_\mu\varphi_0(i\theta\sigma^\mu\bar{\theta}) + \frac{1}{4}\Box\varphi_0(x)\theta\theta\bar{\theta}\bar{\theta}, \text{conj.}\right) \\
=\ &T|_{\varphi_0(x)} + (\partial_\mu\varphi_0{}^i(x)\partial_i T|_{\varphi_0(x)} - \partial_\mu\varphi_0^{*i}(x)\partial_{i^*}T|_{\varphi_0(x)})(i\theta\sigma^\mu\bar{\theta}) \\
+\ &\left(\frac{1}{4}\partial_\mu\varphi_0{}^i(x)\partial^\mu\varphi_0{}^j(x)\partial_i\partial_j T|_{\varphi_0(x)} + \frac{1}{4}\partial_\mu\varphi_0^{*i}(x)\partial^\mu\varphi_0^{*j}(x)\partial_{i^*}\partial_{j^*}T|_{\varphi_0(x)}\right. \\
&\left.-\frac{1}{2}\partial_\mu\varphi_0{}^i(x)\partial^\mu\varphi_0^{*j}(x)\partial_i\partial_{j^*}T|_{\varphi_0(x)}\right)\theta\theta\bar{\theta}\bar{\theta}, \quad (3.19)
\end{aligned}
$$

and the difference between $T|_{\phi_0(y)}$ and $T|_{\phi_0(x)}$ contains at least a term proportional to θ and $\bar{\theta}$. In the next subsection, $\varphi_0(x)$ is regarded as a background field.

3.3   Nonlinear sigma model in the Kähler normal coordinate

We now apply the results of the previous section to background field methods in sigma models. The dynamics are described by quantum fluctuations around a vacuum expectation value, given by

$$
\langle\Phi(y, \theta)\rangle = \langle\varphi(y)\rangle = \varphi_0(y). \quad (3.20)
$$

14

Here we consider a bosonic background and assume that $\langle F \rangle = 0$, so that supersymmetry is unbroken. Note that the background depends on y but not x, as clarified below. The relation with the bosonic background in the ordinary coordinates x is Eq. (3.18). We replace the complex coordinates $z^i - z_0{}^i$ of the Kähler manifolds in the last section with chiral superfields

$$\Delta \Phi^i(y, \theta) \overset{\text{def}}{=} \Phi^i(y, \theta) - \varphi_0{}^i(y). \tag{3.21}$$

As a generalization of Eq. (2.13) to superfields, we perform the coordinate transformation to the Kähler normal coordinates $\xi^i(x, \theta, \bar\theta)$:

$$
\begin{aligned}
\xi^i(y, \theta) &= \xi^i(\Delta\Phi^j(y, \theta), \varphi_0(y)) \\
&= \Delta\Phi^i + \sum_{N=2}^{\infty} \frac{1}{N!} g^{ij^*} K_{, i_1 \cdots i_N j^*} |_{\varphi_0(y)} \Delta\Phi^{i_1} \cdots \Delta\Phi^{i_N} \\
&= \Delta\Phi^i + \frac{1}{2} \Gamma^i{}_{jk}|_{\varphi_0(y)} \Delta\Phi^j \Delta\Phi^k + \frac{1}{6} g^{im^*} \partial_l \Gamma_{m^* jk}|_{\varphi_0(y)} \Delta\Phi^j \Delta\Phi^k \Delta\Phi^l + \cdots. \tag{3.22}
\end{aligned}
$$

Note that the two sets of chiral superfields $\xi^i(x, \theta, \bar\theta)$ and $\Delta\Phi^i(x, \theta, \bar\theta)$ have the same chirality: $\bar D_{\dot\alpha} \Delta\Phi^i = 0$ implies $\bar D_{\dot\alpha} \xi = 0$, as discussed in the previous subsection. This is because the coefficients are evaluated with the *bosonic* background $\Phi(y, \theta) = \varphi_0(y)$.[6] The bosonic and fermionic parts of Eq. (3.22) in x are obtained from Eq. (3.17) to give

$$\varphi_\xi^i(x) = \varphi_{\Delta\Phi}^i(x) + \sum_{N=2}^{\infty} \frac{1}{N!} g^{ij^*} K_{, i_1 \cdots i_N j^*} |_{\varphi_0(x)} \varphi_{\Delta\Phi}^{i_1}(x) \cdots \varphi_{\Delta\Phi}^{i_N}(x), \tag{3.23}$$

$$\psi_\xi^i(x) = \psi_{\Delta\Phi}^i(x) + \sum_{N=1}^{\infty} \frac{1}{N!} g^{ij^*} K_{, i_1 \cdots i_N k j^*} |_{\varphi_0(x)} \varphi_{\Delta\Phi}^{i_1}(x) \cdots \varphi_{\Delta\Phi}^{i_N}(x) \psi_{\Delta\Phi}^k(x), \tag{3.24}$$

respectively. Here we have set $\Delta\Phi(y, \theta) = \varphi_{\Delta\Phi}(y) + \sqrt{2}\theta\psi_{\Delta\Phi}(y) + \theta\theta F_{\Delta\Phi}(y)$ and $\xi(y, \theta) = \varphi_\xi(y) + \sqrt{2}\theta\psi_\xi(y) + \theta\theta F_\xi(y)$.

The same expansion as in Eq. (2.14) is obtained by the transformation (3.22). We thus obtain a Kähler normal coordinate expansion of the Lagrangian of supersymmetric nonlinear sigma models, which is manifestly invariant under the supersymmetry transformation and the general coordinate transformation:

$$\mathcal{L} = \int d^4\theta \sum_{N=4}^{\infty} \sum_{M=2}^{N-2} \frac{1}{M!(N-M)!} K_{, i_1 \cdots i_M j_1^* \cdots j_{N-M}^*} |_{\varphi_0(y)} \xi^{i_1} \cdots \xi^{i_M} \xi^{\dagger j_1} \cdots \xi^{\dagger j_{N-M}},$$

---

[6] If we consider background *superfields* Φ_0, two sets of superfields cannot be simultaneously chiral, since $K_{, i_1 \cdots i_N j_1^*}(\Phi_0, \Phi_0{}^\dagger)$ possesses chiral and anti-chiral superfields [9].

where the covariance of the coefficients is ensured by the theorem in the previous section.

The Lagrangian in terms of the component fields can be calculated in the same way as Eq. (3.11), by noting that coefficients are not constant and we must calculate a product of Eq. (3.19) with $T = K,_{i_1 \cdots j_1^* \cdots}|_{\varphi_0(y)}$ and Eq. (3.5) before integration over θ. Instead, we can integrate over θ first, and then transform to the Kähler normal coordinates at the level of the component fields.[7] To do this, we must calculate

$$\partial_\mu \varphi^i = \partial_\mu(\varphi_0^i + \Delta \varphi^i) = \partial_\mu \varphi_0^i + \partial_\mu \left(\varphi_\xi^i - \frac{1}{2}\Gamma^i{}_{jk}|_{\varphi_0} \varphi_\xi^j \varphi_\xi^k + \cdots \right), \qquad (3.26)$$

where the inverse transformation of Eq. (3.23) is needed.

In the case of a constant background, $\partial \varphi_0 = 0$, the integration over θ in (3.25) can be performed easily. At the component level, we do not need Eq. (3.26). An expansion to sixth order can be obtained by substituting Eqs. (2.30), (2.32) and (2.33) into Eq. (3.11) as (we omit the subscript ξ)

$$\begin{aligned}
\mathcal{L} ={}& g_{ij^*}(\varphi, \varphi^*)\partial_\mu \varphi^i \partial^\mu \varphi^{*j} + i g_{ij^*}(\varphi, \varphi^*)\bar{\psi}^j \bar{\sigma}^\mu \partial_\mu \psi^i + i\Gamma_{j^* ik}(\varphi, \varphi^*)\partial_\mu \varphi^k \bar{\psi}^j \bar{\sigma}^\mu \psi^i \\
&+ \frac{1}{4}R_{ij^*kl^*}(\varphi, \varphi^*)\psi^i \psi^k \bar{\psi}^j \bar{\psi}^l, \\
={}& g_{ij^*}|\partial_\mu \varphi^i \partial^\mu \varphi^{*j} + i g_{ij^*}|\bar{\psi}^j \bar{\sigma}^\mu \partial_\mu \psi^i \\
&+ \left[R_{ij^*kl^*}|\varphi^k \varphi^{*l} + \frac{1}{2}D_m R_{ij^*kl^*}|\varphi^m \varphi^k \varphi^{*l} + \frac{1}{2}D_{m^*}R_{ij^*kl^*}|\varphi^k \varphi^{*l}\varphi^{*m} \right. \\
&+ \frac{1}{6}D_n D_m R_{ij^*kl^*}|\varphi^n \varphi^m \varphi^k \varphi^{*l} + \frac{1}{6}D_{n^*}D_{m^*}R_{ij^*kl^*}|\varphi^k \varphi^{*l}\varphi^{*m}\varphi^{*n} \\
&+ \left. \frac{1}{12}(D_{(n^*}D_m R_{ij^*kl^*)_{\mathrm{ah}}} + g^{or^*}R_{o(j^*ml^*}R_{in^*k)r^*})|\varphi^m \varphi^k \varphi^{*l}\varphi^{*n} \right] \\
&\times (\partial_\mu \varphi^i \partial^\mu \varphi^{*j} + i\bar{\psi}^j \bar{\sigma}^\mu \partial_\mu \psi^i) \\
&+ i\left[R_{ij^*kl^*}|\varphi^{*l} + D_m R_{ij^*kl^*}|\varphi^m \varphi^{*l} + \frac{1}{2}D_{m^*}R_{ij^*kl^*}|\varphi^{*l}\varphi^{*m} \right. \\
&+ \frac{1}{2}D_n D_m R_{ij^*kl^*}|\varphi^n \varphi^m \varphi^{*l} + \frac{1}{6}D_{n^*}D_{m^*}R_{ij^*kl^*}|\varphi^{*l}\varphi^{*m}\varphi^{*n} \\
&+ \left. \frac{1}{6}(D_{(n^*}D_m R_{ij^*kl^*)_{\mathrm{ah}}} + g^{or^*}R_{o(j^*ml^*}R_{in^*k)r^*})|\varphi^m \varphi^{*l}\varphi^{*n} \right] \partial_\mu \varphi^k \bar{\psi}^j \bar{\sigma}^\mu \psi^i
\end{aligned}$$

---

[7] This is ensured by the fact that field redefinitions of chiral superfields reduce to field redefinitions of components fields of bosons and fermions, as seen in Eq. (3.17).

$$+\frac{1}{4}\left[R_{ij^*kl^*}| + D_m R_{ij^*kl^*}|\varphi^m + D_{m^*}R_{ij^*kl^*}|\varphi^{*m}\right.$$

$$+\frac{1}{2}D_n D_m R_{ij^*kl^*}|\varphi^n\varphi^m + \frac{1}{2}D_{n^*}D_{m^*}R_{ij^*kl^*}|\varphi^{*m}\varphi^{*n}$$

$$+\frac{1}{3}(D_{(n^*}D_m R_{ij^*kl^*)_{\mathrm{ah}}} + g^{or^*}R_{o(j^*ml^*}R_{in^*k)r^*} - g^{or^*}R_{oj^*ml^*}R_{in^*kr^*})|\varphi^m\varphi^{*n}\Big]$$

$$\times\psi^i\psi^k\bar{\psi}^j\bar{\psi}^l + O(7). \tag{3.27}$$

The first two terms are motion terms of the bosons and the fermions and the others are interaction terms.

We can obtain low-energy theorems of scattering amplitudes to $O(p^2)$ (two derivative order) by using the above expression. The low-energy scattering amplitudes for two bosons can be calculated by summing up tree graphs of the fourth order interactions. One can obtain the low-energy theorems expressed in terms of the curvature tensor of a Kähler manifold, since the fourth order term of the bosons is the curvature tensor [7]. A calculation of many-body scattering amplitudes requires expansions to higher orders. One can obtain low-energy theorems expressed in terms of the curvature tensor and the covariant derivatives.

Acknowledgements

A   Geometry of Kähler manifolds

In this appendix, we explain the minimum of Kähler manifolds. (For details see, e.g., Ref. [10].) A Kähler manifold is defined as a complex manifold equipped with a Hermitian metric and the Kähler condition ($d\Omega = 0$ where $\Omega = ig_{ij^*}dz^i \wedge dz^{*j}$). As a result of the Kähler condition, the metric can be expressed in terms of a Kähler potential as

$$g_{ij^*}(z, z^*) = \frac{\partial^2 K(z, z^*)}{\partial z^i \partial z^{*j}}, \tag{A.1}$$

at least in a coordinate patch.[8] The connection with mixed indices disappears as a result of the compatibility condition of the complex structure, $DJ = 0$. The non-zero connections are given by

$$\Gamma^k_{\ ij} = g^{kl^*}\frac{\partial g_{jl^*}}{\partial z^i} = g^{kl^*}K,_{ijl^*} \tag{A.2}$$

and their conjugates. Derivatives of the metric are

$$g_{ij^*,k} = \frac{\partial g_{ij^*}}{\partial z^k} = g_{mj^*}\Gamma^m_{\ ik} = g_{kj^*,i} \overset{\text{def}}{=} \Gamma_{j^*ik},$$

$$g_{ij^*,k^*} = \frac{\partial g_{ij^*}}{\partial z^{*k}} = g_{im^*}\Gamma^{m^*}_{\ j^*k^*} = g_{ik^*,j^*} \overset{\text{def}}{=} \Gamma_{ij^*k^*}. \tag{A.3}$$

Independent components of the curvature tensor are

$$R^{i^*}_{\ j^*kl^*} = \partial_k(g^{mi^*}g_{mj^*,l^*}) \tag{A.4}$$

and their conjugates. We use the curvature tensor with lower indices:

$$
\begin{aligned}
R_{ij^*kl^*} & \overset{\text{def}}{=} g_{im^*}R^{m^*}_{\ j^*kl^*} = g_{ml^*}\frac{\partial \Gamma^m_{\ ik}}{\partial z^{*j}} = \frac{\partial^2 g_{kl^*}}{\partial z^i \partial z^{*j}} - g^{mn^*}\frac{\partial g_{ml^*}}{\partial z^{*j}}\frac{\partial g_{kn^*}}{\partial z^i} \\
& = K,_{ij^*kl^*} - g^{mn^*}K,_{mj^*l^*}K,_{n^*ik} .
\end{aligned}
\tag{A.5}
$$

The curvature tensor has the symmetry

$$R_{ABCD} = -R_{ABDC} = -R_{BACD} = R_{CDAB}, \tag{A.6}$$

$$R_{ij^*kl^*} = R_{kj^*il^*} = R_{il^*kj^*}, \tag{A.7}$$

where the uppercase Roman letters are used for both holomorphic and anti-holomorphic indices. The second identity is a result of the Kähler condition.

B A proof of the theorem

In this section we prove the theorem. The starting point is Eq. (2.14). We use the normal coordinates ω, and all differentiations are with respect to ω in this section. Before giving a proof of the theorem, we prove a lemma.

---

[8] To define the metric consistently on the whole Kähler manifold, the Kähler potentials in the union of two different patches are related as $K'(z', z'^*) = K(z, z^*) + g(z) + g^*(z^*)$, where g is a function. This is also called a Kähler transformation, like Eq. (2.2).

We denote (a set of) the Kähler potential differentiated at most n times as $K_{,(n)}$. $(K_{,(n)} \subset K_{,(n+1)}.)$ For example, $K_{,ijk^*} \in K_{,(3)}$. Note that, although all terms with $(n,1)$ and $(1,n)$ indices, $K_{,i_1 \cdots i_n j_1^*}$ and $K_{,ij_1^* \cdots j_n^*}$, vanish at the origin $(\omega = 0)$, $K_{,i_1 \cdots i_n j_1^*}| = K_{,ij_1^* \cdots j_n^*}| = 0$, they do not disappear at an arbitrary value of ω in general. (For example, the connection (2.32) vanishes only at the origin.) The theorem states that the remaining $K_{,(N)}$ become covariant tensors *at the origin*.

If we fix the ordering of the holomorphic and anti-holomorphic covariant derivatives, there is a one-to-one correspondence between the $(M, N-M)$ tensor

$$R^{(N)}{}_{j_1^* \cdots j_{N-M-2}^* i_1 \cdots i_{M-2} i j^* k l^*} \stackrel{\text{def}}{=} D_{j_1^*} \cdots D_{j_{N-M-2}^*} D_{i_1} \cdots D_{i_{M-2}} R_{ij^*kl^*}$$

and the coordinate derivative of the Kähler potential $K_{,j_1^* \cdots j_{N-M-2}^* i_1 \cdots i_{M-2} i j^* k l^*} \in K_{,(N)}$. (For example, see Eqs. (A.5), (2.19) and (2.23) for the first few orders.) By generalizing these equations, we obtain the following lemma giving a relation between the covariant $(M, N-M)$ tensor and the coordinate derivative of the Kähler potential.

Lemma. The curvature tensor covariantly differentiated $(M-2, N-M-2)$ times can be written as $(N \geq 4,\ 2 \leq M \leq N-2)$

$$R^{(N)}{}_{j_1^* \cdots j_{N-M-2}^* i_1 \cdots i_{M-2} i j^* k l^*}$$

$$\in K_{,j_1^* \cdots j_{N-M-2}^* i_1 \cdots i_{M-2} i j^* k l^*} + \sum_{\alpha=1}^{N-3} (-1)^\alpha (g^{-1})^\alpha \underbrace{K_{,(N-1)} \cdots K_{,(N-1)}}_{(\alpha+1)\text{-times}}. \quad \text{(B.1)}$$

The first term in the second line is an element of $K_{,(N)}$, and all terms have M holomorphic and $(N-M)$ anti-holomorphic indices. The g^{-1} in the second term denotes the inverse metric g^{ij^*}. Each g^{-1} contracts indices of two different $K_{,(N-1)}$'s. (Proof) We use mathematical induction for the proof.

1. First, we show that Eq. (B.1) holds for $N = 4$:
   $R^{(4)}{}_{ij^*kl^*} = R_{ij^*kl^*} = K_{,ij^*kl^*} - g^{mn^*} K_{,mj^*l^*} K_{,ikn^*}$ is in the form of Eq. (B.1). (This is trivial when N is less than 4.)

2. Second, we show Eq. (B.1) for $(N + 1)$-th order, assuming that it holds at N-th order. One of elements at $(N + 1)$-th orders is

$$R^{(N+1)}{}_{j_1^* \cdots j_{N-M-1}^* i_1 \cdots i_{M-2} i j^* k l^*} = D_{j_{N-M-1}^*} R^{(N)}{}_{j_1^* \cdots j_{N-M-2}^* i_1 \cdots i_{M-2} i j^* k l^*}$$

$$= \partial_{j_{N-M-1}^*} R^{(N)}{}_{j_1^* \cdots j_{N-M-2}^* i_1 \cdots i_{M-2} i j^* k l^*}$$

$$- \sum_{a=1}^{N-M-2} \Gamma^{m^*}{}_{j_{N-M-1}^* j_a^*} R^{(N)}{}_{j_1^* \cdots j_{a-1}^* m^* j_{a+1}^* \cdots j_{N-M-2}^* i_1 \cdots i_{M-2} i j^* k l^*}$$

$$- \Gamma^{m^*}{}_{j_{N-M+1}^* j^*} R^{(N)}{}_{j_1^* \cdots j_{N-M-2}^* i_1 \cdots i_{M-2} i m^* k l^*}$$

$$- \Gamma^{m^*}{}_{j_{N-M+1}^* l^*} R^{(N)}{}_{j_1^* \cdots j_{N-M-2}^* i_1 \cdots i_{M-2} i j^* k m^*}. \tag{B.2}$$

Since $\Gamma^i{}_{jk} = g^{il^*} K_{,jkl^*} \in g^{-1} K_{,(3)}$, the last three terms are of the form $-\sum g^{-1} K_{,(3)} R^{(N)}$. Moreover, this can be rewritten in the form

$$\sum_{\alpha=1}^{N-2} (-1)^\alpha (g^{-1})^\alpha \underbrace{K_{,(N)} \cdots K_{,(N)}}_{(\alpha+1)\text{-times}} , \tag{B.3}$$

since $K_{,(3)} \subset K_{,(N-1)} \subset K_{,(N)}$. The first term on the right-hand side of Eq. (B.2) has the form

$$\partial_{j_{N-M-1}^*} K_{,j_1^* \cdots j_{N-M-2}^* i_1 \cdots i_{M-2} i j^* k l^*}$$

$$+ \sum_{\alpha=1}^{N-3} (-1)^\alpha \left[-(g^{-1})^{\alpha+1} K_{,(3)} \underbrace{K_{,(N-1)} \cdots K_{,(N-1)}}_{(\alpha+1)\text{-times}} + (g^{-1})^\alpha \underbrace{K_{,(N)} \cdots K_{,(N)}}_{(\alpha+1)\text{-times}} \right], \tag{B.4}$$

where we have used $\partial g^{-1} = -(g^{-1})^2 K_{,(3)}$ from Eq. (2.21). The second term on the right-hand side is also of the form of Eq. (B.3). We thus have proved

$$R^{(N+1)}{}_{j_1^* \cdots j_{N-M-1}^* i_1 \cdots i_{M-2} i j^* k l^*}$$

$$\in K_{,j_1^* \cdots j_{N-M-1}^* i_1 \cdots i_{M-2} i j^* k l^*} + \sum_{\alpha=1}^{N-2} (-1)^\alpha (g^{-1})^\alpha \underbrace{K_{,(N)} \cdots K_{,(N)}}_{(\alpha+1)\text{-times}}. \tag{B.5}$$

The right-hand side has the same form as Eq. (B.1), where the first term is an element of $K_{,(N+1)}$.

We also have to show the lemma for another element of $(N+1)$-th order, $R^{(N+1)}{}_{j_1^* \cdots j_{N-M-2}^* i_1 \cdots i_{M-1} i j^* k l^*}$. The difference between this standard ordering and the non-standard ordering $D_{j_{M-1}} R^{(N)}{}_{j_1^* \cdots j_{N-M-2}^* i_1 \cdots i_{M-2} i j^* k l^*}$ can be written as products of the curvature tensor and terms from $R^{(4)}$ to $R^{(N)}$ as a result of Eq. (2.26). Since we can show that the latter can be written in the form of Eq. (B.1), as in the same manner above, the former can be also written in the form of Eq. (B.1).

3. From 1 and 2, we prove (B.1) for any $N(\geq 4)$. (Q.E.D.)

Now we are ready to give a proof of the theorem. We would like to show that all coefficients in the expansion (2.14) $K_{,i_1 \cdots i_M j_1^* \cdots j_{N-M}^*}\,|$ *evaluated at the origin* $\omega = 0$ can be expressed by covariant quantities *at the origin*. Now, we have the relation Eq. (B.1) between these coefficients and covariant tensors. The left-hand side is, of course, covariant. Our task is to show that each term on the right-hand side evaluated at the origin is also covariant.

(Proof of the theorem.) We again use mathematical induction for the proof. Note that $K_{,(3)}\,|$ vanishes, by our choice of the Kähler normal coordinates.

1. By the equation $R_{ij^*kl^*} = K_{,ij^*kl^*} - g^{mn^*} K_{,mj^*l^*} K_{,ikn^*}$, we can show that $K_{,(4)}\,| \ni K_{,ij^*kl^*}\,| = R_{ij^*kl^*}|$ is covariant. ($K_{,(2)} \ni g_{ij^*}$ is covariant at any ω.)

2. We assume that all terms of order less than $(N+1)$, $K_{,(4)} \cdots K_{,(N)}$, are covariant *at the origin*. (Terms with $(n,1)$ and $(1,n)$ indices disappear at the origin.) Then $K_{,(N+1)}\,|$ is also expressed in terms of covariant quantities, by Eq. (B.5), since all other terms are covariant, by assumption.

3. From 1 and 2, all of the coefficients in Eq. (2.14) are covariant. (Q.E.D.)

References

[1] L. Alvarez-Gaumé, D. Z. Freedman and S. Mukhi, Ann. of Phys. **134** (1981) 85.

[2] B. Zumino, Phys. Lett. **87B** (1979) 203.
L. Alvarez-Gaumé and D. Z. Freedman, Comm. Math. Phys. **80** (1981) 443.

[3] T. E. Clark and S. T. Love, Nucl. Phys. **B301** (1988) 439.

[4] An integral representation of the metric expansion in Riemann normal coordinates is discussed in U. Müller, C. Schubert and A. E. M. van de Ven, J. Geo. Rel. Grav. **31** (1999) 1759, gr-qc/9712092. For the method using the geodesics, see references therein.

[5] K. Higashijima and M. Nitta, Prog. Theor. Phys. **103** (2000) 635, hep-th/9911139; Prog. Theor. Phys. **103** (2000) 833, hep-th/9911225; *Supersymmetric Nonlinear Sigma Models*, hep-th/0006025, to appear in Proceedings of Confinement 2000 held at Osaka, Japan, March, 2000; *Auxiliary Field Formulation of Supersymmetric Nonlinear Sigma Models*, hep-th/0008240, to appear in Proceedings of ICHEP 2000 held at Osaka, Japan, July, 2000.
K. Higashijima, T. Kimura, M. Nitta and M. Tsuzuki, Prog. Theor. Phys. **105** (2001) 261, hep-th/0010272.

[6] M. Bando, T. Kuramoto, T. Maskawa and S. Uehara, Phys. Lett. **138B** (1984) 94; Prog. Theor. Phys. **72** (1984) 313; Prog. Theor. Phys. **72** (1984) 1207.
M. Bando, T. Kugo and K. Yamawaki, Phys. Rep. **164** (1988) 217.

[7] K. Higashijima, M. Nitta, K. Ohta and N. Ohta, Prog. Theor. Phys. **98** (1997) 1165, hep-th/9706219.
K. Higashijima and M. Nitta, *Geometry and the Low Energy Theorem in $N = 1$ Supersymmetric Theories*, hep-th/0006038.

[8] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton Univ. Press, Princeton (1992).

[9] B. Spence, Nucl. Phys. **B260** (1985) 531.

[10] M. Nakahara, *Geometry, Topology and Physics*, Institute of Physics Publishing, Bristol and Philadelphia (1990).